Relativistic kicked rotor

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Transport properties in the relativistic analog of the periodically kicked rotor are contrasted under classically and quantum mechanical dynamics. The quantum rotor is treated by solving the Dirac equation in the presence of the time-periodic δ-function potential resulting in a relativistic quantum mapping describing the evolution of the wave function. The transition from the quantum suppression behavior seen in the nonrelativistic limit to agreement between quantum and classical analyses in the relativistic regime is discussed. The absence of quantum resonances in the relativistic case is also addressed.

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I. INTRODUCTION

The study of regular and chaotic motion in periodically driven dynamical systems has been the subject of extensive investigation in both theoretical and experimental contexts [1–7]. The effects of quantization on such systems has also been explored simultaneously with applications in atomic, molecular, optical, and mesoscopic systems. Quantization suppresses classical chaos and the mechanisms for this provide different experimental signatures in the quantum treatment as compared to the classical case. Among these is the so-called dynamical localization phenomenon which is analogous to Anderson localization in solid-state physics [3,8]. A convenient paradigm for illustrating this and other effects of suppression is the δ-kicked rotor, which can now be realized experimentally as well [9]. The classical dynamics of the kicked rotor is described by the well-known standard map [1,6]. This map greatly facilitates the qualitative treatment of the system. Given the temporal separation of kinetic and potential terms, the quantum dynamics can also be represented by a mapping describing the evolution of the wave function [2].

It should be noted that despite the great progress made in our understanding of classical chaos and its manifestations (or lack thereof) in quantum dynamics, the systems studied have been primarily nonrelativistic. However, the work that does exist in the literature on relativistic systems suggests that the modified dynamics could add some interesting twists in both classical and quantum contexts. At first glance, relativistic systems appear "more nonlinear" than their nonrelativistic counterparts due largely to the fact that the relativistic Hamiltonian can be written as the sum of a nonrelativistic Hamiltonian and an effective potential. Thus, the equations of motion written in action-angle variables become more nonlinear than those in the nonrelativistic limit. As we shall illustrate later, this is not true in general. In addition, the study of relativistic dynamics is relevant to a variety of systems in nuclear and particle physics, cosmology, and atomic and plasma physics.

A recent time-dependent example where relativistic dynamical chaos may be exhibited is the case of relativistic electrons in a plasma accelerated by lasers [10]. These so-called wakefield or beat wave accelerators are the subject of extensive theoretical as well as experimental study [10–13]. The charged particle motion in these accelerators is periodically driven by the ponderomotive potential [14], and can be treated in terms of a relativistic kicked particle model. The inclusion of the role of quantum effects in such an acceleration could also be important though that is less clear at present.

With this as a broad motivation, we explore features of both classical and quantum dynamics of the relativistic kicked rotor. The classical analysis is performed using the relativistic standard map for which detailed phase space analysis already exists [15,16]. Our emphasis is on the larger scale transport properties as embodied in the time dependence of the classical relativistic rotor. The presence of the relativistic factor now makes it a two-parameter system which is known to exhibit richer structure in terms of transport kinetics. This further motivates the treatment of the quantum version with the aim of isolating transitions in behavior.

The quantum analysis of the Dirac equation proceeds along the same lines as in the pioneering work of Casati *et al.* [2]. In our case, the result is a quantum mapping describing the evolution of the four-component spinor wave function of the kicked rotor. Note that the quantization of the free relativistic rotor has been considered [17] and the following three limiting cases defined: (i) an elementary limit in which the rotor behaves as an elementary (pointlike) particle, (ii) a classical limit in which it coincides with the classical relativistic rotor and, (iii) a nonrelativistic limit in which it coincides with the nonrelativistic quantum rotor. The first of these is a trivial limit which is implicitly satisfied in this work though we will comment on the other two cases in some detail.

We begin by analyzing the transport in the classical relativistic rotor in the space of two parameters, the kicking strength and the relativistic parameter. This is intended to provide a contrast for the quantum dynamics. The quantum analysis begins (for completeness) with a solution of the Dirac equation for the free rotor eigenvalues and eigenfunctions. The addition of a periodic drive is then assessed, resulting in a quantum map describing the evolution of the

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wave function. This mapping is then used to compute the evolution of the probability density as well as the time dependence of the energy. The last two sections of the paper discuss the comparison of classical and quantum evolutions and the conclusions reached.

II. THE CLASSICAL RELATIVISTIC KICKED ROTOR

As is well known [1,3], the phase space evolution of the nonrelativistic classical kicked rotor is described by the nonrelativistic standard map. A single parameter measuring the strength of the kicking term governs the dynamics which, beyond a well-defined threshold, shows no barriers to unbounded motion in momentum. The resulting diffusion in energy is one of the quantities suppressed on quantizing the dynamics. The relativistic generalization of the kicked rotor problem can also be treated using the corresponding relativistic standard map which has been studied extensively on the basis of phase space analysis including phase space structure as well as resonance overlap [15,16]. By contrast, we are interested in the time dependence of the energy and transport properties of the classical relativistic kicked rotor dynamics.

The relativistic extension of the classical standard map can be obtained [15] by considering the motion of the relativistic electron in the field of an electrostatic wave packet given by

$$E(x,t) = E_0 \sum_{n=-\infty}^{\infty} \sin(kx - n\omega t). \tag{1}$$

The equations of motion for this system have the same form as those for the relativistic δ-kicked classical rotor:

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{pc^2}{\sqrt{m_0^2c^4 + p^2c^2}}, \tag{2}$$

$$\begin{aligned}\frac{dp}{dt} &= -\frac{\partial H}{\partial x} = -eE_0 \sum_{n=-\infty}^{\infty} \sin(kx - n\omega t)\\ &= \frac{2\pi}{\omega} eE_0 \sin(kx) \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{2n\pi}{\omega}\right),\end{aligned} \tag{3}$$

Integrating these equations over one kick results in the relativistic generalization of the standard map [15]

$$P_{n+1} = P_n - \frac{K}{2\pi}\sin(2\pi X_n), \tag{4}$$

$$X_{n+1} = X_n + \frac{P_{n+1}}{\sqrt{1 + \beta^2 P_{n+1}^2}}, \tag{5}$$

where

$$K = \frac{4\pi^2 eE_0 k}{m_0\omega^2}, \quad \beta = \frac{\omega}{kc}. \tag{6}$$

As is seen from Eqs. (4) and (5) the relativistic standard map is a two-parameter map involving K and β, where β is defined as a relativistic factor based on the group velocity ω/k.

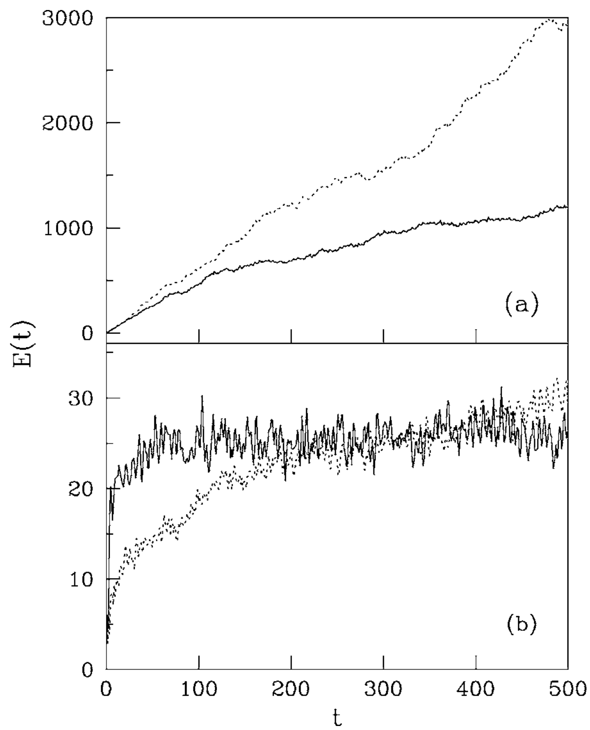

FIG. 1. Time dependence of the classical relativistic kicked rotor energy for weak relativistic (a) and ultrarelativistic (b) regimes. In (a) the two curves correspond to β=0.01 (solid) and 0.000 01 (dashed), respectively. In (b) the solid line corresponds to β=0.1 while the dashed line deals with the resonance case (β=1/2π). In all cases, K=5.0.

It is clear that this map reduces to the nonrelativistic standard map in the limit $\beta \to 0$ whereas the limit $\beta \to 1$ is the ultrarelativistic limit.

Two-parameter maps, in general, display a richer variety of behavior. Of particular interest is the well-known result that boundaries to transport can reform with variation of the two parameters, resulting in parameter windows of bounded and unbounded motion. Earlier work [15,16] discussed aspects of this trend in terms of phase space whereas we will consider the transport behavior to assess the full extent of this behavior. In particular, it is clear from the map that as the momentum grows, the system becomes more integrable for any nonzero value of β. Of course, this occurs at smaller values of momentum as β gets larger. Thus, in the relativistic standard map, energy growth is limited by the presence of invariant curves at higher momenta.

The transport properties are computed by starting from a line in momentum space, iterating the classical map and constructing the phase space averaged energy. Figure 1 shows the time-dependent variation of the energy for a range of β values. Figure 1(a) considers the weakly relativistic regime while Fig. 1(b) deals with the ultrarelativistic regime. Figure 1 makes clear that the diffusion of the energy is greatly suppressed as β increases.

Resonances exist in the relativistic standard map at special values [15] of β=1/2πN where the energy growth takes considerably longer to saturate. This is what is shown by the dashed line in Fig. 1(b) where the continued growth of the

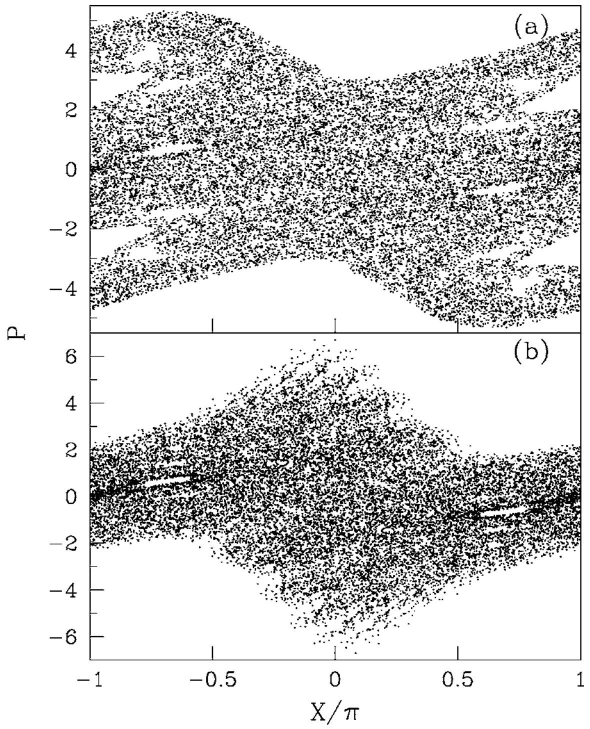

FIG. 2. Phase portraits of the relativistic kicked rotor: (a) nonresonant ($\beta=0.1$) and (b) resonant ($\beta=1/2\pi$) parameter values. As before, $K=5.0$.

energy is seen. By contrast, it is highly suppressed when the value of $\beta=0.1$ which is less than $\beta=1/2\pi$. However, the diffusion is not unlimited even for the resonance case and ultimately saturates for longer times.

Thus the suppression of the energy growth in the classical relativistic kicked rotor depends on the value of the parameter β. For smaller values of β which are close to 0 the growth of the energy is linear and suppression occurs only at very long times. At intermediate β values, the energy grows (not necessarily linearly) for some time, beyond which it saturates. For higher values still (starting from around $\beta=0.1$) and excluding resonance cases the growth of the energy is quickly suppressed. The differences between resonant and nonresonant scenarios is clear from the differences in phase space as seen in Fig. 2. Despite the larger value of β, the dynamics reaches higher values of momentum in the case of resonance, though it ultimately saturates.

As noted earlier, the relativistic standard map is a two-parameter map which suggests the existence of a richer variety of behavior as the two parameters are varied. In order to illustrate this feature, we consider the transport exponent α defined by the long-time variation of energy with time, i.e., $E(t)\sim t^{\alpha}$ for large t. As is clear from our discussion thus far, this exponent clearly varies with both K and β. We compute this exponent for a range of values in K-β space and plot the exponent as a function of the two parameters. As seen from Fig. 3, this provides a clear snapshot of the changes in kinetics.

The small-β regime (note the logarithmic scale) is dominated by diffusive behavior though there are clear lines of suppression even for the smallest values of β where nonrel-

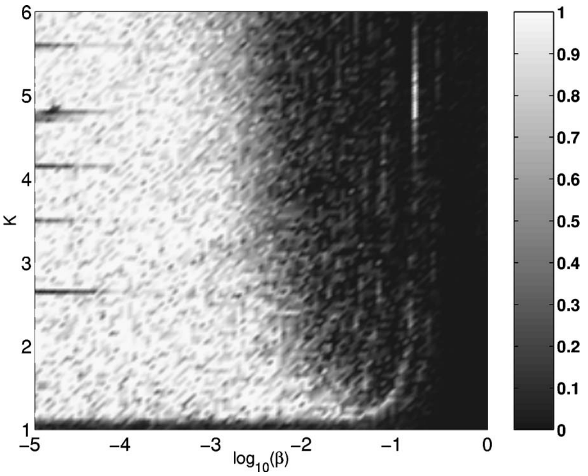

FIG. 3. Behavior of the classical kicked rotor in the K-β plane. For each value, the transport exponent α where $E(t)\propto t^{\alpha}$ is plotted. Thus the extremes of 0 and 1 correspond to saturation and diffusion, respectively.

ativistic behavior is anticipated. This is the result of accelerator modes in the nonrelativistic standard map ($\beta=0$) which quickly drive the iterates of the mapping to higher momenta, where integrable dynamics sets in. Thus, we encounter the peculiar phenomenon that the accelerator modes in the nonrelativistic standard map, of which the higher-order ones are hard to compute, are more clearly visible in the relativistic version of the same mapping.

The transition boundary between diffusion and saturation ($\alpha=0$), for fixed K and changing β, varies with K. In the regime where $\alpha=0$, there appear narrow vertical windows of diffusive growth in energy. These correspond to the classical resonances mentioned earlier. Having provided a synopsis of the variability of classical transport in the two-parameter space, we now proceed to our primary focus of quantum treatment in the same system.

III. THE DIRAC EQUATION FOR THE FREE ROTOR

The solution for the nonrelativistic quantum kicked rotor is written as a basis expansion in terms of the free rotor states and we proceed in an analogous manner in the relativistic problem. Though the nonrelativistic solutions are readily available in almost any graduate textbook [18], the same is not true for the relativistic counterpart. As such, for completeness, we begin by considering the solutions of the Dirac equation for the free rotor and the explicit forms of eigenvalues and eigenfunctions. We note that despite the fact that the quantization of the relativistic free rotor has been considered before [17,19–21], the solution of the Dirac equation has not yet been obtained in explicit form.

Thus the wave equation we want to solve is

$$(\vec{\alpha}\cdot\vec{p}+\beta)\Phi = E\Phi, \tag{7}$$

where $\alpha=\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}$, $\beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}$ are the usual Dirac matrices; $p=i\partial/\partial\theta$ is the rotor momentum, θ its angular position, and $\Phi=\begin{pmatrix}\varphi\\ \chi\end{pmatrix}$. Note that the system of units $m=\hbar=c=1$ is used throughout.

So, for the components φ and χ we have

$$\vec{\sigma}\cdot\vec{p}\chi+\varphi=E\varphi,\qquad \vec{\sigma}\cdot\vec{p}\varphi-\chi=E\chi. \tag{8}$$

Expressing χ in terms of φ we obtain the second-order differential equation [22,23]

$$p^2\varphi=(E^2-1)\varphi \tag{9}$$

or

$$-\frac{1}{2}\frac{\partial^2\varphi}{\partial\theta^2}=\frac{1}{2}(E^2-1)\varphi=\varepsilon\varphi \tag{10}$$

where

$$\varepsilon=\frac{1}{2}(E^2-1). \tag{11}$$

Solving this equation we can construct the solution of Eq. (7):

$$\Phi_n=N_n\begin{pmatrix}\varphi_n\\ \chi_n\end{pmatrix}=N_n\begin{pmatrix}0\\ 0\\ \dfrac{in}{0+E_n}e^{-i\theta}\\ -\dfrac{in}{0+E_n}e^{i\theta}\end{pmatrix}e^{in\theta}, \tag{12}$$

with normalization constant

$$N_n=\frac{1}{\sqrt{2\pi[1+n^2/(1+E_n)^2]}} \tag{13}$$

and energy eigenvalues

$$E_n=\sqrt{1+n^2}. \tag{14}$$

It is clear from Eqs. (12) and (11) that in the limit $n\ll 1$ or, in the usual system of units, $n\ll mc^2$ (which corresponds to small rotation energies) both the energy and wave function go over to their nonrelativistic counterparts.

IV. QUANTUM RELATIVISTIC KICKED ROTOR

We can now proceed to the solution of the relativistic quantum kicked rotor problem, where the Dirac equation now has the form

$$i\frac{\partial\psi}{\partial t}=\hat{H}\psi \tag{15}$$

where

$$\hat{H}=\alpha_\theta p_\theta+\beta+\varepsilon_0\delta_T(t)\cos\theta \tag{16}$$

and

$$\delta_T(t)=\sum_{m=-\infty}^{\infty}\delta(t-mT)$$

with T being the period of the kicks. The method to be used for the solution of this equation is the same as in Ref. [2]. We expand the wave function $\psi(\theta,t)$ in terms of the relativistic free rotor eigenfunctions

$$\psi(\theta,t)=\sum_n A_n(t)\Phi_n. \tag{17}$$

Over any period T between δ- the function kicks the coefficients $A_n(t)$ evolve freely as

$$A_n(t+T)=A_n(t)e^{-iE_nT}, \tag{18}$$

where in order to have a control parameter to govern the transition from relativistic to nonrelativistic regimes, we rewrite the free rotor energy as

$$E_n=\beta^{-2}(\sqrt{1+\beta^2n^2}-1),$$

with $\beta=c^{-2}$ and $m=\hbar=1$.

During the infinitesimal time interval of a kick Eq. (15) takes the form

$$i\frac{\partial\psi}{\partial t}=-\varepsilon_0\cos\theta\delta_T(t)\psi, \tag{19}$$

and the wave functions immediately before and immediately after [corresponding to times $(t+T)$ and $(t+T^+)$, respectively] the kick are related by

$$\psi(\theta,t+T^+)=\psi(\theta,t+T)e^{iK\cos\theta}, \tag{20}$$

where $K=\varepsilon_0T$. Expanding both sides of this equation in terms of the relativistic free rotor eigenfunctions, we have

$$\sum_n A_n(t+T^+)\Phi_n(\theta)=\sum_{r,s}A_r(t+T)\Phi_r(\theta)b_s(K)e^{is\theta}. \tag{21}$$

Here we have used the expression

$$e^{iK\cos\theta}=\sum_s b_s(K)e^{is\theta}, \tag{22}$$

where

$$b_s(K)=i^sJ_s(K)=b_{-s}(K), \tag{23}$$

where J_s are ordinary Bessel functions of the first kind.

Multiplying both sides of Eq. (21) and taking into account that

$$\int_0^{2\pi}d\theta\,\Phi_n^*\Phi_re^{is\theta}=4\pi N_rN_n\left(1+\frac{nr}{(1+E_n)(1+E_r)}\right)\delta_{n,r+s} \tag{24}$$

one obtains the relativistic quantum mapping

$$A_n(t+T^+)=2\pi N_n\sum_r N_rA_r(t)b_{n-r}(K)\times\left(1+\frac{nr}{(1+E_n)(1+E_r)}\right)e^{-iE_rT}, \tag{25}$$

where N_r and N_n are given by Eq. (13). Once again, it is evident that in the limit $n,r\ll 1$ this relativistic quantum map coincides with the nonrelativistic one.

It is immediately apparent from the mapping that one aspect of the nonrelativistic quantum rotor will not appear in

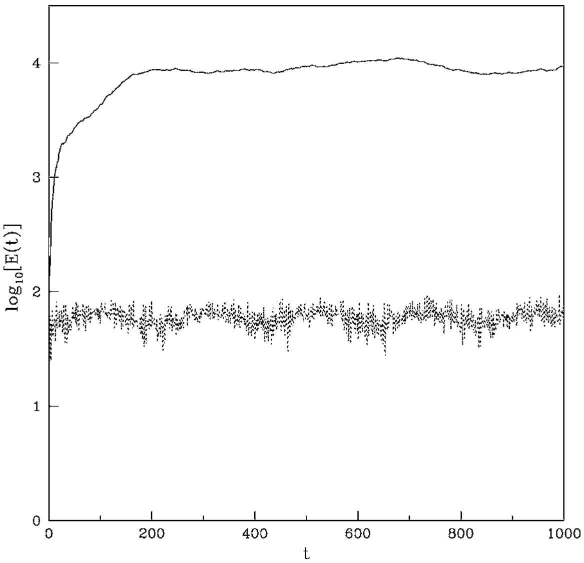

FIG. 4. Time dependence of the quantum relativistic kicked in the weak relativistic (solid line, β=0.0001) and ultrarelativistic (dashed line, β=0.1) regimes. For both cases K=10.

the relativistic case. The nonrelativistic quantum rotor exhibits quantum resonances corresponding to special values of the kicking period T. These are situations where the free evolution does not randomize the phases at which the kicks occur and, as a consequence, the resulting wave function is extended [3]. These resonances do not appear in the quantum relativistic case as seen from the structure of the mapping (21). It follows simply from the fact that E_n are irrational numbers. This means that there can be no unlimited growth of energy in the relativistic case.

A key feature of the nonrelativistic quantum kicked rotor is the suppression of the diffusive growth of energy seen in the classical dynamics. In that case, the suppression is due to phase randomization between kicks and the wave function is localized in a manner analogous to the Anderson model for the metal-insulator transition [8]. As we have seen from the earlier results, the energy growth saturates even in the classical relativistic case, leading to the expectation of strong localization in the relativistic quantum problem.

The time dependence of the energy can be calculated as

$$\langle E \rangle = \sum_n E_n \rho(n,t), \tag{26}$$

where $\rho(n,t)=|A_n(t)|^2$.

In Fig. 4 we contrast the time dependence of the quantum relativistic kicked energy for representative values of β corresponding to different regimes. As one can see from this figure, for smaller values of β the behavior is similar to that for the nonrelativistic quantum rotor. However, the saturation occurs much earlier in the relativistic case though we can no longer interpret this behavior as the suppression of classical diffusion.

This feature is illustrated in Fig. 5 where classical and quantum evolutions (in terms of energy dependence on time) are contrasted for the strongly relativistic (β=0.1) case.

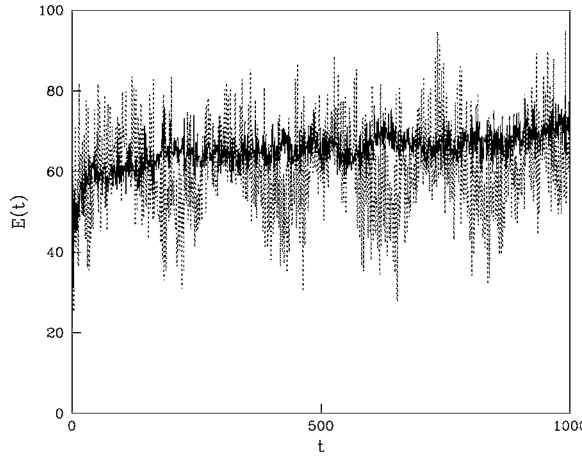

FIG. 5. Time dependences of quantum (dashed line) and classical (solid line) rotor energies for $\beta=0.1, K=15.0$.

Aside from fluctuations in the quantum case, there is good agreement between the quantum and classical energy values resulting from the near integrability of the dynamics. The value of β considered was sufficiently large that saturation occurs in a few kicks.

In the case of the nonrelativistic quantum kicked rotor localization properties can be quantified by the participation ratio defined as $\zeta=\Sigma_n\rho(n,t)^2/\Sigma_n|A_n(t)|^4$ taken at large times. ζ varies as K^2 in the nonrelativistic case when all other parameters are held fixed. By contrast, the relativistic treatment exhibits near independence on the parameter K. This is shown in Fig. 6 where the variation of ζ with respect to the parameters K and β is plotted. The K^2 variation for small β is seen while the relativistic regime ($\beta>0.1$) shows a much weaker dependence on K, if any at all. For the parameter range shown, a plot of the classical exponent α defined earlier shows a clear separation between diffusive and saturated behavior. The boundary in the classical case coincides well with the transition to K independence in the quantum parameter ζ.

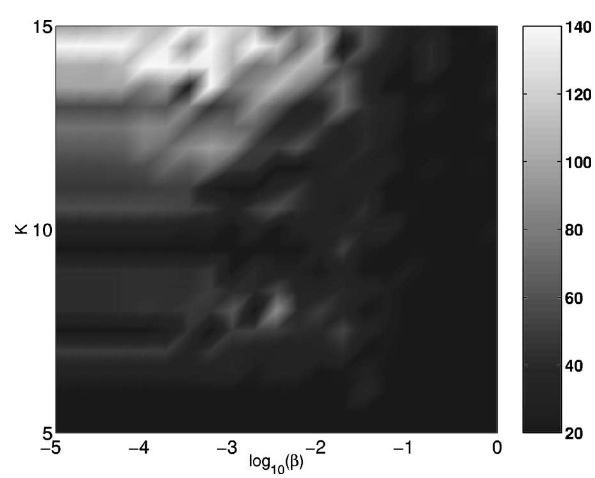

FIG. 6. Variation of participation ratio ζ in the two-parameter space.

V. CONCLUSIONS

In summary, we have assessed aspects of both classical and quantum dynamics for the relativistic kicked rotor with a focus on the behavior of energy with time and possible suppression in the quantum dynamics. We find that even in the classical dynamics, the relativistic case shows saturation in energy growth due to the presence of barriers to transport in the classical phase space. This occurs even in the case of classical resonances at β values which are inverse multiples of 2π though the time of saturation can be considerably longer. An interesting effect was shown for K values for which accelerator modes exist in the nonrelativistic limit. At these values, any nonzero value of β leads to saturation after a short time because the classical iterates are driven to higher momenta quickly. Thus, in a two-parameter diagram of the transport exponent, higher-order accelerator modes in the nonrelativistic map are clearly identified by parameter windows of zero exponent in the midst of otherwise diffusive behavior.

The quantum dynamics of the relativistic kicked rotor is studied in terms of a mapping derived from the Dirac equation. The structure of the mapping immediately suggests the absence of quantum resonances in the relativistic case. Another important difference comes from the fact that, in the relativistic case, agreement between classical and quantum dynamics proceeds as for a integrable system. Thus, from the standpoint of correspondence, the relativistic case is far less interesting.

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